K25P 1135

Reg. No. : ....

Name : ....

IV Semester M.Sc. Degree (C.B.S.S. – Supple./Imp.) Examination, April 2025
(2021 and 2022 Admissions)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Let X be a normed space and $A \in BL(X)$. Then prove that A is invertible if and only if A is bounded below and surjective.

2. Let X and Y be normed spaces. Let F_1 and F_2 be in BL(X, Y) and $k \in K$. Then prove that

   $(F_1 + F_2)' = F'_1 + F'_2$ and $(kF_1)' = kF'_1$.

3. If $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$ in a normed space X, then show that $x_n + y_n \xrightarrow{w} x + y$.

4. Let X and Y be normed spaces and $F : X \to Y$ be linear. Prove that F is a compact map if and only if for every bounded sequence (x_n) in X, $(F(x_n))$ has a subsequence which converges in Y.

5. Let H be a Hilbert space and $A \in BL(H)$. Then prove that $Z(A) = R(A^*)^{\perp}$ and $Z(A^*) = R(A)^{\perp}$.

6. Define numerical range of an operator on a Hilbert space and prove or disprove that it is a closed subset of K.

P.T.O.

PART – B

Answer **any four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Then prove that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

   b) Let X be a Banach space over K and $A \in BL(X)$. Let $k \in K$ such that $|k|^p > ||A^p||$, for some positive integer p. Then prove that $k \notin \sigma(A)$ and

   $$(A - kI)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{k^{n+1}}.$$

8. a) Let X be a nonzero Banach space over C and $A \in BL(X)$. Then prove that

   $$r_\sigma(A) = \inf_{n=1,2,\ldots} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

   b) Let X be a normed space and X_0 be a dense subspace of X. For $x' \in X'$, let $F(x')$ denote the restriction of x' to X_0. Then prove that the map F is a linear isometry from X' onto X'_0.

9. a) Let X be a normed space and (x_n) be a sequence in X. Then prove that (x_n) is weak convergent in X if and only if (i) (x_n) is a bounded sequence in X and (ii) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x' in some subset of X' whose span is dense in X'.

   b) Let X be a separable normed space. Then prove that every bounded sequence in X' has a weak* convergent subsequence.

Unit – II

10. Let X be a normed space. Then prove that X is reflexive if and only if every bounded sequence in X has a weak convergent subsequence.

11. a) Let X and Y be normed spaces and $F : X \to Y$ be linear. If F is continuous and of finite rank, then prove that F is a compact map and R(F) is closed in Y. Conversely, prove that if X and Y are Banach spaces, F is a compact map and R(F) is closed in Y, then F is continuous and of finite rank.

    b) Let X and Y be normed spaces and $F : X \to Y$ be linear. Let X be reflexive and $F(x_n) \to F(x)$ in Y whenever $x_n \xrightarrow{w} x$ in X. Then prove that $F \in CL(X, Y)$.

12. a) Let X be a normed space, $A \in CL(X)$ and $0 \neq k \in K$. If (x_n) is a bounded sequence in X such that $A(x_n) - kx_n \to y$ in X, then prove that there is a subsequence (x_{n_j}) of (x_n) such that $x_{n_j} \to x$ in X and $A(x) - kx = y$.

b) Let X be a normed space and $A : X \to X$. Let $0 \neq k \in K$ and Y be a proper closed subspace of X such that $(A - kI)(X) \subset Y$. Then prove that there is some $x \in X$ such that $||x|| = 1$ and for all $y \in Y$,

$$\|A(x) - A(y)\| \geq \frac{|k|}{2}.$$

Unit – III

13. a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$,

$\langle A(x), y\rangle = \langle x, B(y)\rangle$.

b) Let H be a Hilbert space and $A \in BL(H)$. Prove the following :

i) $R(A) = H$ if and only if A^* is bounded below.

ii) $R(A^*) = H$ if and only if A is bounded below.

14. a) Let H be a Hilbert space and $A \in BL(H)$. Let A be self-adjoint. Then prove that

$||A|| = \sup\{|\langle A(x), x\rangle| : x \in H, ||x|| \leq 1\}$

and $A = 0$ if and only if $\langle A(x), x\rangle = 0$ for all $x \in H$.

b) Let $A \in BL(H)$ be self-adjoint. Then prove that A or $-A$ is a positive operator if and only if $|\langle A(x), y\rangle|^2 \leq \langle A(x), x\rangle \langle A(y), y\rangle$ for all $x, y \in H$.

15. a) Let $H \neq \{0\}$ and $A \in BL(H)$ be self-adjoint. Then prove that

$\{m_A, M_A\} \subset \sigma_a(A) = \sigma(A) \subset [m_A, M_A]$.

b) Let $H \neq \{0\}$. Let $A \in BL(H)$ be self-adjoint. Then prove that

$||A|| = \max\{|m_A|, |M_A|\} = \sup\{|k| : k \in \sigma(A)\}$.

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